exaggerated magnitude relative to that of Eq. (17). For kinematically nonlinear analysis of beams this effect certainly would be noticeable, especially for large deflection problems. It should be obvious that $\bar{\kappa}$ has *no* physical meaning for the problem of the bending of a beam in terms of the coordinate x as defined here. The reason for this is that the coordinate x is along the original position of the beam, as is typical in mechanics. With the present definition of x, Eq. (18) does not account for the well-known effective shortening due to transverse deflections. This effective shortening generates a deflection u even in an inextensional beam, due to a transverse deflection v. If the derivatives in Eq. (18) are redefined to be with respect to the shortened axis $\bar{x}=x+u$, then it is not difficult to show that

$$\kappa=\frac{\mathrm{d}^2 v/\mathrm{d}\bar{x}^2}{[1+(\mathrm{d}v/\mathrm{d}\bar{x})^2]^{3/2}} \tag{19}$$

Thus, the key factor in making the familiar expression for curvature compatible with the kinematics of planar beam deformation is that the independent variable in such an expression must be the actual distance of the material point from the root of the beam (i.e., $\bar{x}$). It should be noted that description of beam kinematics in terms of $\bar{x}$ is often inconvenient since the limits of integration over the beam are functions of the deformation. This is obviously not the case when x is used as the coordinate.

Concluding Remarks

In this Note, a simplified example was chosen deliberately to show that the physical curvature for planar beam deformation is not the same as the commonly known expression found in elementary calculus texts. Even this simplified example proves to be complex relative to linear beam analyses. That pitfalls exist even in the best attempts to adapt textbook expressions to fit one's particular analytical needs should be obvious from this development. Clearly, the safest approach for any nonlinear analysis, including one involving geometric nonlinearity, should be a careful one based on a sound application of fundamental principles.

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Application of the Generalized Inverse in Structural System Identification

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A PRINCIPAL objective of system identification is to improve the mathematical model of a test structure that can then be used to predict the system dynamics and the effects of the modification of the structure. Analytically, the structure is discretized and the mass matrix M_a and stiffness matrix K_a can be obtained using the finite element method. Of course, these matrices are only approximate expressions to the structure. On the other hand, vibration testing provides incomplete information on the modal properties of the actual structure. Two mathematical methods have been applied to the improvement of analytical models. Assuming the measured natural frequencies and mode shapes to be exact, Berman[1] and Berman and Wei[2] used the Lagrange multiplier method to minimize a matrix norm introduced by Baruch and Bar Itzhack.[3] The improved mass and stiffness matrices thus obtained exactly predict the measured natural frequencies and mode shapes. Alternatively, the generalized inverse method was used by Rodden[4] in computing the stiffness matrix, by Berman and Flannelly[5] in identifying the mass matrix, and by Chen and Garba[6] in the parameter estimation. However, these three papers dealt only with linear equations without constraints. In this Note, application of the generalized inverse to constrained systems is presented. The basic assumption is that, as in Refs. 1 and 2, the measured modal properties are exact. It is found that the present approach is physically sound and algebraically simpler in formulation.

Generalized Inverse

Penrose[7] showed that for any matrix $A(p\times q)$ there is a unique matrix $A^+(q\times p)$ satisfying the four equations

$$AA^+A=A,\quad A^+AA^+=A^+,\quad (AA^+)^*=AA^+,$$
$$(A^+A)^*=A^+A \tag{1}$$

where the asterisk denotes the conjugate transpose of a matrix and A^+ is known as the Moore-Penrose generalized inverse. Specifically, if A is real, so is A^+; if A is nonsingular, then $A^+=A^{-1}$. Note also that, if A is of rank q, the A^+ is the only left inverse of A having rows in the row space of A^*. In this case, $A^+=(A^*A)^{-1}A^*$. Conversely, if A is of rank p, then A^+ is the only right inverse of A having columns in the column space of A^* and $A^+=A^*(AA^*)^{-1}$. (See Ref. 8.)

The generalized inverse techniques have been used extensively in solving the linear equation

$$AX=B \tag{2}$$

and the general solution reads

$$X=A^+B+(I-A^+A)Y \tag{3}$$

in which Y is arbitrary. It is obvious that $(I-A^+A)Y$ satisfies the associated equation $AX=0$ and is referred to as the homogeneous solution. The particular solution A^+B is the minimum-norm least-square solution of Eq. (2). That is,

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denoting $\|A\| = \text{trace}\,(A^*A)$,

$$\|A^+B\| \leq \|X\| \text{ and } \|AA^+B - B\| = \min_X \|AX - B\|$$

for all solution X (see Ref. 9). The geometrical interpretation of Eq. (3) is as follows. The particular solution is the projection of X on the column space of A^* and the homogeneous solution its projection on the orthogonal complement of that space. In other words, they are perpendicular to each other. Therefore, we have $\|X\| = \|A^+B\| + \|(I - A^+A)Y\|$.

Application to System Identification

All matrices used hereafter are real and thus the conjugate transpose in the above section should be replaced by the transpose. Let M_a $(n \times n)$ and M $(n \times n)$ be analytical and improved mass matrices. The orthogonality condition of the normalized modes $\Phi^T M^\Phi = I$ can be rewritten as

$$\Phi^T W_1^{-1} (W_1 \Delta M W_1) W_1^{-1} \Phi = I - \Phi^T M_a \Phi \tag{4}$$

in which $\Delta M = M - M_a$ and $W_1 (n \times n)$ is any symmetric, nonsingular weighting matrix. Since the number of measured modes m is, in general, less than the number of degrees of freedom n, the measured modal matrix $\Phi(n \times m)$ is rectangular with full column rank m. Applying Eq. (3) twice to minimize $\|W_1 \Delta M W_1\|$ yields

$$W_1 \Delta M W_1 = U(I - \Phi^T M_a \Phi) U^T \tag{5}$$

where

$$U = (\Phi^T W_1^{-1})^+ = (W_1^{-1}\Phi)(\Phi^T W_1^{-2}\Phi)^{-1}$$

Equation (5) does not include the homogeneous solution because no other conditions must be satisfied. This is evident if we note that the improved mass matrix

$$M = M_a + W_1^{-1} U(I - \Phi^T M_a \Phi) U^T W_1^{-1} \tag{6}$$

is symmetric.

To identify the stiffness matrix, we start with the eigenequation

$$\Phi^T K = \Omega^2 \Phi^T M \tag{7}$$

and the constraint conditions

$$K^T = K \tag{8}$$

Define $\Delta K = K - K_a$ and $R = \Omega^2 \Phi^T M - \Phi^T K_a$. Since the analytical stiffness matrix K_a is symmetric, Eq. (8) implies that ΔK is symmetric. Postmultiplying Eq. (7) by a symmetric, nonsingular weighting matrix W_2 leads to

$$\Phi^T W_2^{-1} (W_2 \Delta K W_2) = R W_2 \tag{9}$$

The general solution of Eq. (9) is

$$W_2 \Delta K W_2 = QRW_2 + (I - QP)Y \tag{10}$$

where $P = \Phi^T W_2^{-1}$ and $Q = P^+$. Using the symmetry of ΔK, we get from Eq. (10)

$$Y^T(I - P^T Q^T) + W_2 R^T Q^T = (I - QP)Y + QRW_2 \tag{11}$$

Since $P(I - QP)Y = (P - P)Y = 0$, premultiplying Eq. (11) by P and rearranging yields

$$Y^T = QP(QRW_2 - W_2 R^T Q^T)(I - P^T Q^T)^+ \tag{12}$$

As noted earlier, the particular solution and the homogeneous solution in Eq. (10) are perpendicular; thus, minimizing $\|W_2 \Delta K W_2\|$ implies minimizing $\|(I - QP)Y\|$ or equivalently $\|Y\|$. Thus, the solution of Eq. (12) is given by

$$(I - QP)Y = (W_2 R^T Q^T - QRW_2) P^T Q^T \tag{13}$$

The improved stiffness matrix is obtained by substituting Eq. (13) into Eq. (10)

$$K = K_a + W_2^{-1} QR + R^T Q^T W_2^{-1} - W_2^{-1} QRW_2 P^T Q^T W_2^{-1} \tag{14}$$

In particular, when $W_1 = M_a^{-1/2}$ and $W_2 = M^{-1/2}$, Eqs. (6) and (14) reduce to

$$M = M_a + M_a\Phi(\Phi^T M_a \Phi)^{-1}(I - \Phi^T M_a \Phi)(\Phi^T M_a \Phi)^{-1}\Phi^T M_a \tag{15}$$

$$K = K_a + M\Phi(\Phi^T K_a \Phi + \Omega^2)\Phi^T M - (K_a \Phi\Phi^T M + M\Phi\Phi^T K_a) \tag{16}$$

which are the same as Berman's results.[1,2]

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The Fuel Property/Flame Radiation Relationship for Gas Turbine Combustors

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Introduction

TWO recently published papers[1,2] reported gas turbine combustor flame radiation data, and gave different correlations of these data with two fuel properties: weight percent hydrogen (H) and weight percent polycyclic

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